2.1 – Determinants by Cofactor Expansion

Definition: If A is a square matrix, then the **minor of entry a_{ij}** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry a_{ij}**.

$A^{-1} = \frac{1}{ad-bc}[\text{stuff}]$

$(-1)^{i+j}$

$A = [a_{ij}]$

Example: Find all the minors and cofactors of $A = \begin{bmatrix} \overset{+}{2} & \overset{-}{3} & \overset{+}{-7} \\ \overset{-}{4} & \overset{+}{-2} & \overset{-}{-9} \\ \overset{+}{2} & \overset{-}{5} & \overset{+}{6} \end{bmatrix}$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is a determinant. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$M_{11} = \begin{vmatrix} -2 & -9 \\ 5 & 6 \end{vmatrix} = -12 + 45 = 33$ $\quad M_{12} = \begin{vmatrix} 4 & -9 \\ 2 & 6 \end{vmatrix} = 42$ $\quad M_{13} = \begin{vmatrix} 4 & -2 \\ 2 & 5 \end{vmatrix} = 24$

$C_{11} = 33$ $\quad C_{12} = -42$ $\quad C_{13} = 24$

$M_{21} = \begin{vmatrix} 3 & -7 \\ 5 & 6 \end{vmatrix} = 53$ $\quad M_{22} = \begin{vmatrix} 2 & -7 \\ 2 & 6 \end{vmatrix} = 26$ $\quad M_{23} = \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = 4$

$C_{21} = -53$ $\quad C_{22} = 26$ $\quad C_{23} = -4$

$M_{31} = \begin{vmatrix} 3 & -7 \\ -2 & -9 \end{vmatrix} = -41$ $\quad M_{32} = \begin{vmatrix} 2 & -7 \\ 4 & -9 \end{vmatrix} = 10$ $\quad M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -16$

$C_{31} = -41$ $\quad C_{32} = -10$ $\quad C_{33} = -16$

Definition: If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **cofactor expansions of A**. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} \text{ (cofactor expansion along the } j\text{th column)}$$
$$= a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \text{ (cofactor expansion along the } i\text{th row)}$$

Example: Compute the determinant of the matrix A above.

$$\begin{array}{cccccc} & + & & - & & + \\ \det(A) & 2M_{11} & - & 3M_{12} & - & 7M_{13} \end{array}$$

$$\det(A) = 2(33) - 3(42) - 7(24) = -228$$

$$2\times 2: \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

#11 Use the arrow technique of Figure 2.1.1 to evaluate the determinant.

$$\begin{array}{ccc} 20 & +84 & +6 \end{array}$$

$$\left|\begin{matrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{matrix}\right| \begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = 45 - 110 = -65$$

$$-20 - 7 + 72$$

$$-(ceg + fha + ibd)$$

why? $\left|\begin{matrix} a & d & g \\ b & e & h \\ c & f & i \end{matrix}\right| \begin{matrix} a & d \\ b & e \\ c & f \end{matrix}$

$$aei + dhc + bgf$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} \text{ \& stuff...}$$

$$a(ei - hf)$$
$$\hookrightarrow aei - ahf$$

Theorem 2.1.1 If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

, lambda

#16 Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{vmatrix} = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix}$$

$$\det(A) = 0 \Rightarrow (\lambda-4)(\lambda^2-\lambda-6) = 0$$

$$(\lambda-4)(\lambda-3)(\lambda+2) = 0$$

$$\lambda = -2, 3, 4$$

Theorem 2.1.2 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

#31 Evaluate the determinant of the given matrix by inspection.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 6$$

$$1\begin{vmatrix} 1 & -4 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{vmatrix} - 2\underbrace{\begin{vmatrix} 0 & -4 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{vmatrix}}_{=0} + 7\underbrace{\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{vmatrix}}_{=0} + 3\underbrace{\begin{vmatrix} 0 & 1 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix}}_{=0}$$

$1(1)(2)(3)$ eventually